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Pareto-optimal security strategies in matrix games with fuzzy payoffs

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Abstract

We present a new methodology for the analysis of fuzzy payoff matrix games. The main difficulty that appears in the study of these games is the comparison between the payoff values associated to the strategies of the players because these payoffs are fuzzy quantities. Our approach does not transform the fuzzy payoffs to crisp numbers via standard defuzzyfication but we use standard fuzzy orders which allows us to find solutions within the same space of fuzzy numbers. Moreover, we provide a method to solve these games finding equivalent fuzzy linear programs whose maximal solutions give the solutions of the games.

Keywords: Two-person game; Fuzzy payoffs; Pareto-optimal security strategies; Fuzzy linear programming

1. Introduction

In the standard theory of zero-sum games the payoffs are known with certainty [32]. However, in the real world the certainty assumption is not realistic in many occasions. This lack of precision may be modeled by different ways and one can find in the literature several approaches to deal with this vagueness:

- In some cases one can assume some probability distribution on the data giving rise to the so-called stochastic payoffs games as in [15,36,37].
- At times the vagueness is modeled via the payoffs which are assumed to range on k-dimension vector spaces or more generally on general lattices [5,14,16,17].
- Finally, one can also model impreciseness via fuzzy logic [40]. In these cases, payoffs are represented by fuzzy numbers, see, e.g. [1,3,11,31,38].

The class of games with fuzzy payoffs models and analyzes actual competitive or cooperative situations that present some source of vagueness and impreciseness on any of its elements.

We will consider the case of two-person games in which although the players have perfectly defined their strategies, they have lack of precision on the knowledge of the associated payoffs. So this class of fuzzy two-person games, will be called fuzzy payoff matrix games (FPMG).

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Starting with the seminal studies by Aubin [1,2] different approaches have appeared in the literature of fuzzy games. Campos [6], Campos et al. [7] and Bector et al. [4] deal with a class of fuzzy two-person games defuzzifying the fuzzy numbers that appear in the formulation applying Yager's ranking function. Some different papers even simplify the analysis considering only particular families of numbers, as for instance in Maeda [29], who reduces himself to consider triangular fuzzy numbers. Other authors as Chen and Larbani [8] permit player I to choose a unique α-cut that will be applied to all the fuzzy numbers of the payoff matrix, whereas player II chooses one crisp value of these cuts that will be used in the final numeric payoff matrix. Therefore, they also transform the original fuzzy game to a standard scalar zero-sum game. One can also find in the literature papers analyzing bimatrix games and existence of equilibria in a fuzzy environment, as for instance in [23,25,26]. The interested reader is also referred to [27] for a comprehensive survey on non-cooperative fuzzy games in normal form.

In those situations where there is not a total order among the payoffs, comparing the payoff obtained by the players is much more difficult than comparing them in scalar games, thus classical solution concepts are not applicable. For this reason, new solution concepts have been proposed in recent years. Particularly, the concept of *P*areto-*O*ptimal *S*ecurity *S*trategy (POSS) becomes very important in order to solve these classes of games (see [19,20,14] and the references therein).

In this paper we deal with fuzzy payoff two-person matrix games where each component of the payoff matrix is a general fuzzy number, i.e. not restricted to belong to any particular family, see Dubois and Prade [10,12]. As for the comparison between fuzzy numbers we assume standard fuzzy orders introduced by Ramik and Římánek [34] and González and Vila [21,22], which avoid reducing these fuzzy elements to real numbers thus allowing richer indifference relationships among fuzzy entities.

Using these orders leads us to consider Pareto-optimal security strategies (POSS) as solution concept for the class of fuzzy payoff two-person matrix games. POSS is an extension of the concept of equilibrium based on the security levels of the players in the game. This solution concept shares some of the most important properties listed in the all approaches to this subject in the literature:

- POSS strategies are obtained as maximal solutions of a fuzzy linear problem associated to the game based on a special partial ordering.
- Payoffs of both players are fuzzy values and there is a weak duality relation between them.

The goal of this paper is to prove the above properties of FPMG. To this end, Section 2 introduces and recalls some basic definitions needed throughout the paper. In particular, we recall the definition of standard fuzzy order [21,22]. Here, we also give the formal definition of fuzzy payoff matrix games and propose a natural way to find maximal solutions of fuzzy linear problems under standard fuzzy orders. Section 3 establishes the relationship between fuzzy payoff matrix games and fuzzy linear programs. Then applying the results in the previous section, we provide a method to solve the class of fuzzy games considered in the paper: finding POSS strategies of these games can be done solving multiobjective linear problems. Finally, Section 4 proves a duality relation, in the spirit of [39], between the payoff values obtained by players I and II in this class of games.

2. Model and basic concept

In this section we introduce the class of games that we will deal with. Our goal is to find strategies solving these two-person zero-sum games. Therefore, this implies to compare the payoffs obtained by the two players. However, these payoffs are fuzzy numbers, thus first of all we will have to agree on the partial order used to perform these comparisons.

Definition 1. A fuzzy number $\widetilde{a}$ is a fuzzy set on the space of real numbers $\mathbb{R}$, whose membership function $\mu_{\widetilde{a}} : \mathbb{R} \to [0, 1]$ satisfies the following conditions:

1. there is a real number c, such that $\mu_{\widetilde{a}}(c) = 1$,
2. $\mu_{\widetilde{a}}$ is upper semicontinuous,
3. $\mu_{\widetilde{a}}$ is quasi-concave,
4. $supp(\widetilde{a})$ is compact, where $supp(\widetilde{a})$ denotes the support of $\widetilde{a}$.

We denote the set of all fuzzy numbers by $\mathbb{N}(\mathbb{R})$.

Let $\widetilde{a}$ be any fuzzy number and let $\alpha \in (0, 1]$ be a real number. The set $a_\alpha = \{x \in \mathbb{R}/\mu_a(x) \geq \alpha\}$, is called the α-cut set of $\widetilde{a}$. For $\alpha = 0$ we set $a_0 = cl\{x \in \mathbb{R}/\mu_a(x) > 0\}$, where *cl* denotes the closure of sets. Since α-cuts of a fuzzy number $\widetilde{a}$ are closed intervals, we denote the α-cut set of $\widetilde{a}$ by $[a^1_\alpha, a^2_\alpha]$, where $a^1_\alpha = \inf(a_\alpha)$ and $a^2_\alpha = \sup(a_\alpha)$.

Comparing fuzzy numbers is a milestone in the field of fuzzy programming. In spite of there is not a total agreement on which order should be universally used in these comparisons, one can find several approaches to address this question in the literature (see, e.g. [9,13,18,24,30]). Each one of them gives rise to a different paradigm. In this paper we will follow a proposal of ordering previously considered by Gonzalez and Vila [21,22] and Ramik and Římánek [34]. Our choice is motivated by the fact that these orderings do not necessarily induce complete orders on the set of fuzzy numbers, because they do not map fuzzy numbers onto the reals but onto k-dimension spaces with $k > 1$. This fact leads to enhance indifference relationships [21]. Throughout the paper we will refer to these orders as *standard fuzzy orders*.

The rationale behind these orderings rests on approximating any exact fuzzy number by a finite number of α-cuts. This means to discretize the original numbers. Although in general this implies a loss of information, in most cases considered in the literature (particularly in all fuzzy numbers induced by piecewise linear membership functions) this approximation is exact. Indeed, if the membership function is given by a piecewise linear function then it has a finite number of pieces and therefore one needs only a finite number of different α-cut sets to exactly describe the corresponding fuzzy number. This way, one can compare '*exactly*' fuzzy numbers by comparing finite number of α-cut sets. In addition, since piecewise quasi-concave linear functions are dense in the set of quasi-concave functions this approach can be used to approximate fuzzy numbers within any given accuracy. From now on we denote a generic set of cuts by $\Upsilon = \{\alpha_1, \ldots, \alpha_r\} \subset [0, 1]$, with $\alpha_1 < \alpha_2 < \cdots < \alpha_{r-1} < \alpha_r = 1$.

Definition 2. Let $\Upsilon = \{\alpha_1, \ldots, \alpha_r\}$ be a set of cuts. A standard ranking function is a function $f : \mathbb{N}(\mathbb{R}) \to \mathbb{R}^{r\times 2}$, such that $f(\widetilde{a}) = (p_{ij}(\widetilde{a})) \in \mathbb{R}^{r\times 2}$, where $p_{ij}(\widetilde{a}) = a^j_{\alpha_i},\ i = 1, \ldots, r,\ j = 1, 2$.

Using this function any partial order on $\mathbb{R}^{r\times 2}$ induces a natural ordering on $\mathbb{N}(\mathbb{R})$. We will use the componentwise order on $\mathbb{R}^{r\times 2}$.

Definition 3. A fuzzy order, $\preceq$, is standard if there exists a standard ranking function f such that

$$\widetilde{a} \preceq \widetilde{b} \Leftrightarrow f(\widetilde{a}) \leq f(\widetilde{b}) \quad \text{with } \widetilde{a}, \widetilde{b} \in \mathbb{N}(\mathbb{R}),$$

where $\leq$ is the componentwise order on $\mathbb{R}^{r\times 2}$.

Any fuzzy order defined by a standard ranking function f induces the following equivalence relationship on $\mathbb{N}(\mathbb{R})$:

$$\widetilde{a} \simeq \widetilde{b} \Leftrightarrow f(\widetilde{a}) = f(\widetilde{b}). \tag{1}$$

We will refer to this relationship as the indifference relationship. Therefore, we will simplify the problem considering the classes on $\mathbb{N}(\mathbb{R})/\simeq$. Note that $f(\widetilde{a})$ induces a canonical representative, $\tilde{f}(\widetilde{a})$, of the indifference class to which $\widetilde{a}$ belongs to. Thus, overcharging the notation we will refer to the fuzzy number $\tilde{f}(\widetilde{a})$, for any $\widetilde{a} \in \mathbb{N}(\mathbb{R})$, although we are actually referring to the indifference class that contains $\widetilde{a}$, on the above quotient space.

In general, standard fuzzy orders are partial. This means that the concept of maximality (minimality) of fuzzy numbers on a set must be understood in the sense of the above partial ordering and therefore we must consider sets of maximal (minimal) elements in the order.

Consider the following fuzzy linear programming problem, where the sense of $\preceq -\min$ must be understood as finding the minimal elements by the relationship $\preceq$:

$$\begin{aligned} (FLP)_{\preceq}\ \preceq -\min \quad & \tilde{c}x \\ \text{s.t.} \quad & \tilde{A}x \preceq \tilde{b}, \\ & x \geq 0,\ x \in \mathbb{R}^n, \end{aligned}$$

where $\tilde{c}, \tilde{b}$ are fuzzy vectors, $\tilde{A}$ is a matrix with fuzzy entries, and $\preceq$ is a standard fuzzy order defined by the set of cuts $\Upsilon = \{\alpha_1, \ldots, \alpha_r\}$. This set of cuts gives rise to $p_{ij}(.),\ i = 1, \ldots, r,\ j = 1, 2$.

Our next result states that finding the minimal elements of $(FLP)_{\preceq}$ can be done computing the set of non-dominated solutions of the following multiobjective linear program:

$$
\begin{aligned}
(MLP)_{\leq} \quad \leq -\min \quad & (p_{ij}(\tilde{c})x)_{\substack{i=1\ldots r\\ j=1,2}} \\
\text{s.t.} \quad & p_{ij}(\tilde{A})x \leq p_{ij}(\tilde{b}),\ i=1,\ldots,r,\ j=1,2, \\
& x \geq 0, x \in \mathbb{R}^n,
\end{aligned} \tag{2}
$$

where $p_{ij}(\widetilde{A}) = (p_{ij}(\widetilde{a_{kl}}))_{kl},\ k=1,\ldots,m,\ l=1,\ldots,n.$

Lemma 4. *The set of minimal solutions to* $(FLP)_{\preceq}$ *coincides with the non-dominated solutions of* $(MLP)_{\leq}$.

Proof. Observe that x is a solution of $(FLP)_{\preceq}$ if and only it does not exist y satisfying $\tilde{A}y \preceq \tilde{b}$, $y \geq 0$, $y \in \mathbb{R}^n$ and such that $\tilde{c}y \prec \tilde{c}x$.

Clearly, by Definition 3, the above is equivalent to the fact that it does not exist y satisfying $f(\tilde{A}y) \leq f(\tilde{b})$, $y \geq 0$, $y \in \mathbb{R}^n$ such that $f(\tilde{c}y) \leq f(\tilde{c}x)$, $f(\tilde{c}y) \neq f(\tilde{c}x)$.

Thus, using that $\preceq$ is a standard order and that these orders preserve linearity (see [21, Proposition 6.1]), $(FLP)_{\preceq}$ can be equivalent written as the problem in (2).

Problem (2) is a multiobjective linear program whose solution set is the set of non-dominated (efficient or Pareto-optimal) solutions. The above chain of equivalences proves that the solutions of $(FLP)_{\preceq}$ coincide with those of $(MLP)_{\leq}$. □

Remark 5. It is worth to observe that any MLP can be efficiently solved using the theory of multiobjective linear programming (see, e.g. [33]), and some of the available software packages as ADBASE [35].

By the above lemma we conclude that, provided that $\preceq$ is a standard fuzzy order defined by the set of cuts $\Upsilon = \{\alpha_1, \ldots, \alpha_r\}$, the solutions to $(FLP)_{\preceq}$ are the non-dominated solutions of $(MLP)_{\leq}$, according to the proposed construction.

In the following, we consider two-person zero-sum games in which the players have perfectly defined their strategies but they have lack of precision on the knowledge of the associated payoffs. This class of fuzzy two-person games will be called fuzzy payoff matrix games (FPMG).

Definition 6. A two-person zero-sum matrix game with fuzzy payoffs, Γ, is a triplet $(S_1^n, S_2^m; \tilde{A})$, where S_1^n (respectively, S_2^m) is the strategy space for player I (respectively, Player II), and $\tilde{A} = (\widetilde{a_{lk}})$, $1 \leq l \leq n$, $1 \leq k \leq m$ is the payoff matrix whose entries are fuzzy numbers.

Each element $\widetilde{a_{lk}} \in \mathbb{N}(\mathbb{R})$ of the matrix $\tilde{A}$ is a fuzzy number that informs us about the knowledge that players I and II have on its own payoffs provided that player I (the maximizer) chooses strategy l and player II (the minimizer) chooses strategy k. Without loss of generality and for the sake of simplicity, these fuzzy numbers will be assumed positive.

We denote the sets of all mixed strategies available for players I and II by

$$X = \left\{ x \in \mathbb{R}^n;\ \sum_{l=1}^{n} x_l = 1, x_l \geq 0,\ i=1,\ldots,n \right\},$$

$$Y = \left\{ y \in \mathbb{R}^m;\ \sum_{k=1}^{m} y_k = 1, y_k \geq 0,\ k=1,\ldots,m \right\}.$$

Clearly, the payoff induced whenever players I and II choose the mixed strategies $x \in X$ and $y \in Y$, respectively, is given by

$$\widetilde{v(x,y)} = x^t \tilde{A} y.$$

3. Pareto-optimal security strategies (POSS)

Let $\Gamma_{\preceq} = (X, Y, \tilde{A})$ be a fuzzy payoff two-person zero-sum matrix game, with the standard fuzzy order $\preceq$. Player I (the maximizer) has to find the maximum outcome against any strategy of player II (the minimizer). Thus, for each strategy $x \in X$ of player I, the security level of player I for x is the payoff that can be guaranteed against any response of player II.

Definition 7. The security level of player I for strategy $x \in X$ is a fuzzy number $\widetilde{\underline{v}(x)} \in \mathbb{N}(\mathbb{R})$ such that

$$f(\widetilde{\underline{v}(x)}) = \left(\inf_{y \in Y} p_{ij}(\widetilde{v(x,y)})\right)_{\substack{i=1\ldots,r\\ j=1,2}}. \tag{3}$$

Analogously, we define the security level of player II for strategy $y \in Y$ as a fuzzy number $\widetilde{\overline{v}(y)} \in \mathbb{N}(\mathbb{R})$ such that

$$f(\widetilde{\overline{v}(y)}) = \left(\sup_{x \in X} p_{ij}(\widetilde{v(x,y)})\right)_{\substack{i=1\ldots,r\\ j=1,2}}.$$

Definition 7 is correct as shown in the following proposition.

Proposition 8. *There exists an indifference class of fuzzy numbers* $\widetilde{\underline{v}(x)}$ *that satisfies* (3).

Proof. Let $x \in X$ be a strategy for player I. We will prove that there exists a fuzzy number $\widetilde{b}$ such that

$$f(\widetilde{b}) = \left(\inf_{y \in Y} x^t (\tilde{A})^j_{\alpha_i} y\right)_{\substack{i=1\ldots,r\\ j=1,2}}, \tag{4}$$

where $(\tilde{A})^j_{\alpha_i} := p_{ij}(\widetilde{A})$. For the sake of readability and to simplify the presentation we denote $(v_{ij}(x))_{\substack{i=1\ldots,r\\ j=1,2}} := (\inf_{y \in Y} x^t (\tilde{A})^j_{\alpha_i} y)_{\substack{i=1\ldots,r\\ j=1,2}}$. Observe that the matrix $\widetilde{A}$ satisfies

$$\text{For all} \quad i = 1, \ldots, r, \quad (\tilde{A})^1_{\alpha_i} \le (\tilde{A})^2_{\alpha_i}, \tag{5}$$

$$\text{If } \alpha_i < \alpha_{i'} \text{ then } (\tilde{A})^1_{\alpha_i} \le (\tilde{A})^1_{\alpha_{i'}},\ (\tilde{A})^2_{\alpha_i} \ge (\tilde{A})^2_{\alpha_{i'}}. \tag{6}$$

Therefore, for any $y \in Y$, as $y \ge 0$, (5) and (6) imply, respectively

(3′) $x^t(\tilde{A})^1_{\alpha_i} y \le x^t(\tilde{A})^2_{\alpha_i} y \quad \forall i \Leftrightarrow v_{i1}(x) \le v_{i2}(x) \quad \forall i$

(4′) If $\alpha_i < \alpha_{i'}$ then $x^t(\tilde{A})^1_{\alpha_i} y \le x^t(\tilde{A})^1_{\alpha_{i'}} y$, $x^t(\tilde{A})^2_{\alpha_i} y \ge x^t(\tilde{A})^2_{\alpha_{i'}} y \Leftrightarrow$

$$v_{i1}(x) \le v_{i'1}(x), \quad v_{i2}(x) \ge v_{i'2}(x).$$

Finally, the above relationships guarantee that the function $\mu_{\widetilde{b}}$ defined as

$$\mu_{\widetilde{b}}(x) = \begin{cases} 1 & \text{if } x \in [v_{r1}(x), v_{r2}(x)], \\ \alpha_i & \text{if } x \in [v_{i1}(x), v_{i2}(x)] \setminus [v_{i+1,i}(x), v_{i+1,2}(x)],\ i = r-1, \ldots, 1, \\ 0 & \text{if } x \in \mathbb{R} \setminus [v_{11}(x), v_{12}(x)] \end{cases}$$

is the membership function of a fuzzy number $\widetilde{b}$ that satisfies (4) what concludes the proof. □

The above concept allows us to analyze fuzzy payoffs matrix games under the rationale of worst case behavior of the opponent. Assuming this attitude against risk a player should choose, among all the strategies that he could play, those maximizing its security levels. The meaning of the above concept must be understood as finding maximal elements with respect to the partial ordering induced by the $\preceq$-relationship.

Definition 9. A strategy $x^* \in X$ is a Pareto-optimal security strategy (POSS) of the game $\Gamma_{\preceq}$, using the standard fuzzy order $\preceq$, for player I iff there is no $x \in X$ such that

$$\widetilde{\underline{v}(x^*)} \prec \widetilde{\underline{v}(x)}.$$

Similarly, one can define POSS for player II.

3.1. Computing POSS

It is well-known that linear programming can be used to find the value and the optimal strategies for any scalar two-person zero-sum matrix game. The following theorem proves that fuzzy linear programming can also be applied to find jointly all POSS and its security levels for a player. This emphasizes the similarity existing between both problems.

Let $\Gamma_{\preceq}$ be a fuzzy payoff matrix game with fuzzy matrix $\tilde{A} = (\tilde{a}_{lk})$, $1 \le l \le n$, $1 \le k \le m$, and assume the standard fuzzy order $\preceq$.

Theorem 10. *The strategy x^* is a POSS and $\tilde{v}^*$ is its security level if and only if $(x^*, \tilde{v}^*)$ is a maximal solution of the fuzzy linear problem*

$$\begin{aligned}(FLP_I)_{\preceq} :\preceq -\max \quad & \tilde{v}\\ \text{s.t.} \quad & x^t\tilde{A} \succeq \tilde{v},\\ & x \in X.\end{aligned}$$

Proof. Let x^* be a POSS for $\Gamma_{\preceq}$. This means that there is no $x \in X$ such that

$$\widetilde{\underline{v}(x^*)} \prec \widetilde{\underline{v}(x)}. \tag{7}$$

Proposition 8 gives us the form of the canonical representative of the indifference class of $\widetilde{\underline{v}(x)}$, $\forall x \in X$. Using this representation

$$f(\widetilde{\underline{v}(x)}) = \left(\inf_{y\in Y} x^t(\tilde{A})^j_{\alpha_i} y\right)_{\substack{i=1\ldots r\\ j=1,2}} = \left(\min_k x^t(\tilde{A}^j_{\alpha_i})_{.k}\right)_{\substack{i=1\ldots r\\ j=1,2}}, \quad k = 1,\ldots,m,$$

where $(\tilde{A}^j_{\alpha_i})_{.k} := \tilde{A}^j_{\alpha_i}(0,\ldots,\overset{k}{1},\ldots,0)^t$. Therefore, we can rewrite (7), using the definition of the standard fuzzy order, as

$$\nexists x \in X \quad \text{such that} \quad \left(\min_k x^t(\tilde{A}^j_{\alpha_i})_{.k}\right)_{\substack{i=1\ldots r\\ j=1,2}} \gneq \left(\min_k x^{*t}(\tilde{A}^j_{\alpha_i})_{.k}\right)_{\substack{i=1\ldots r\\ j=1,2}}. \tag{8}$$

Note that (8) states that x^* is an efficient solution of the problem:

$$\begin{aligned}(VLP)_{\le} : \le -\max \quad & \left(\min_k x^t(\tilde{A}^j_{\alpha_i})_{.k}\right)_{\substack{i=1\ldots r\\ j=1,2}}\\ \text{s.t.} \quad & x \in X.\end{aligned}$$

It is clear that $(VLP)_{\le}$ can be rewritten as

$$\begin{aligned}(MLP_I)_{\le} : \ \le -\max \quad & (v_{ij})_{\substack{i=1\ldots r\\ j=1,2}}\\ \text{s.t.} \quad & x^t(\tilde{A}^j_{\alpha_i}) \ge v_{ij}\cdot 1_m, \forall i, j,\\ & x \in X,\end{aligned}$$

where $1_m = (1,1,\ldots,1) \in \mathbb{R}^m$.

Since $\widetilde{A}$ satisfies (5) and (6) (see the proof of Proposition 8), we can apply Lemma 4 to conclude that solving $(MLP_I)_{\preceq}$ is equivalent to solve

$$\begin{aligned} (FLP_I)_{\preceq} : \preceq -\max \quad & \tilde{v} \\ \text{s.t.} \quad & x^t(\tilde{A}) \succeq \tilde{v}, \\ & x \in X, \; \tilde{v} \in \mathbb{N}(\mathbb{R}). \end{aligned}$$

In conclusion, we have proved that if x^* is a POSS and $\widetilde{\underline{v}(x^*)}$ is its security level, then $(x^*, \widetilde{\underline{v}(x^*)})$ is a maximal solution of $(FLP_I)_{\preceq}$.

Now, by reversing arguments we obtain that if $(x^*, \widetilde{v^*})$ is a maximal solution to $(FLP_I)_{\preceq}$, then x^* is a POSS, and $\widetilde{v^*} \simeq \widetilde{\underline{v}(x^*)}$. $\square$

Remark 11. We observe from the proof of the above theorem that any maximal solution $(x^*, \tilde{v}^*)$ of (FLP_I) satisfies that x^* is a POSS and $\tilde{v}^* \simeq \widetilde{\underline{v}(x^*)}$, the canonical representative of the security level associated to the strategy x^*.

Example 12. Consider the fuzzy matrix game with fuzzy payoff matrix

$$\widetilde{A} = \begin{pmatrix} (180, 5, 10) & (156, 6, 2) \\ (90, 10, 10) & (180, 5, 10) \end{pmatrix}$$

given in [4,6], where (a, α, β) is a triangular fuzzy number. Assume that player I wishes to solve the above game using a standard fuzzy order with set of cuts $\Upsilon = \{0, 1\}$. According to Theorem 10 any POSS strategy must be a non-dominated solution to

$$\begin{aligned} (MLP_I)_{\preceq} \max \quad & v_{11}, v_{12}, v_{21} \\ \text{s.t.} \quad & 175x_1 + 80x_2 \geq v_{11}, \\ & 150x_1 + 175x_2 \geq v_{11}, \\ & 190x_1 + 100x_2 \geq v_{12}, \\ & 158x_1 + 190x_2 \geq v_{12}, \\ & 180x_1 + 90x_2 \geq v_{21}, \\ & 156x_1 + 180x_2 \geq v_{21}, \\ & x_1 + x_2 = 1, \quad x_1, x_2 \geq 0. \end{aligned}$$

Remark that by using triangular numbers the cuts for $\alpha_2 = 1$ reduce to degenerate intervals (crisp numbers), whereas the cuts for $\alpha_1 = 0$ are standard intervals.

Using our approach, we have obtained with the software package ADBASE [35], that $x^* = (0.7916, 0.2084)$ is a POSS strategy of player I and its security level is given by the triangular number $\widetilde{v^*} = (161, 6.79, 3.67)$. However, the solution provided by [6] is $x_C^* = (0.77, 0.23)$ and quoting the author, the value of the fuzzy game is '*around* 160.81', whereas the solution given by [4] is $x_B^* = (0.7725, 0.2275)$ and the value of the fuzzy game is, again quoting the author, '*close to* 160.9'. It is worth observing that our analysis provides an exact triangular number as the value of the game while the other two analysis give only subjective verbal expressions for the value.

Remark 13. We observe that our POSS strategies applied to triangular fuzzy numbers give rise to the *'reasonable solutions'* introduced by Li [28] (see also Bector and Chandra [3] for a detailed presentation of such solution concept).

4. Duality

It is well-known that in standard matrix games both players choose their optimal strategies solving a pair of dual linear programs. Therefore, the payoffs of both players are limited to one another by standard duality in linear programming. In this section we extend this duality result for POSS strategies within the class of fuzzy matrix games with standard fuzzy orders (see [39]).

From the results of Section 3, it directly follows that finding all POSS and their security levels for player II one must solve the following *dual* fuzzy linear program:

$$\begin{aligned}(FLP_{II})_{\preceq} : \preceq -\min \quad & \tilde{w} \\ \text{s.t.} \quad & \tilde{A}y \preceq \tilde{w}, \\ & y \in Y,\ \tilde{w} \in \mathbb{N}(\mathbb{R}).\end{aligned}$$

The following theorem proves a duality relationship between fuzzy values of POSS strategies for players I and II in fuzzy matrix games.

Theorem 14. *If* $(x^*, \tilde{v}^*)$ *is a maximal solution to* $(FLP_I)_{\preceq}$ *and* $(y^*, \tilde{w}^*)$ *is a minimal solution to* $(FLP_{II})_{\preceq}$, *then their security levels satisfy the following weak duality relation*:

$$\tilde{v}^* \preceq \tilde{w}^*.$$

Proof. Let $(x^*, \tilde{v}^*)$ be a maximal solution to $(FLP_I)_{\preceq}$. Theorem 10 ensures that $\tilde{v}^*$ belongs to the same indifference class as $\widetilde{\underline{v}(x^*)}$, the security level of x^*. Moreover, $(x^*, f(\widetilde{\underline{v}(x^*)}))$ is a non-dominated solution to $(MLP_I)_{\leq}$. (See the remark after the proof of Theorem 10.)

Now, each component of any non-dominated solution (x', v') of $(MLP_I)_{\leq}$ satisfies that

$$\begin{aligned}v'_{ij} \leq V_{ij} := \max \quad & v_{ij} \\ \text{s.t.} \quad & x^t(\tilde{A})^j_{\alpha_i} \geq v_{ij} \cdot 1_m, \\ & x \in X,\ 1 \leq i \leq r,\ j = 1, 2.\end{aligned} \tag{9}$$

Analogously, starting from $(y^*, \tilde{w}^*)$, we can prove, using a similar argument to that in Theorem 10, that $(y^*, f(\widetilde{\overline{v}(y^*)}))$ is a non-dominated solution to

$$\begin{aligned}(MLP_{II})_{\leq} : \leq -\min \quad & (w_{ij})_{\substack{i=1\ldots r \\ j=1,2}} \\ \text{s.t.} \quad & (\tilde{A}^j_{\alpha_i})y \leq w_{ij} \cdot 1^t_n, \forall i, j, \\ & y \in Y.\end{aligned}$$

Moreover, each component of any non-dominated solution (y', w') of $(MLP_{II})_{\leq}$ clearly satisfies that

$$\begin{aligned}w'_{ij} \geq W_{ij} := \min \quad & w_{ij} \\ \text{s.t.} \quad & (\tilde{A})^j_{\alpha_i} y \leq w_{ij} \cdot 1^t_n, \\ & y \in Y,\ 1 \leq i \leq r,\ j = 1, 2.\end{aligned} \tag{10}$$

Now, by standard duality theory in linear programming between (9) and (10) we have

$$w'_{ij} \geq W_{ij} = V_{ij} \geq v'_{ij} \quad \forall 1 \leq i \leq r,\ j = 1, 2.$$

Therefore, the above relationships imply, by Definition 3, that

$$\widetilde{\overline{v}(y^*)} \succeq \widetilde{\underline{v}(x^*)},$$

and thus

$$\tilde{w}^* \succeq \tilde{v}^*. \quad \square$$

In general, we have not found necessary and sufficient conditions ensuring strong duality between the pair of problems (FLP_I) and (FLP_{II}), apart from those stating that optimal primal and dual solutions sets have non-empty intersection.

Nevertheless, one could derive sufficient conditions based on the finite intersection property of optimal solutions sets for all α-cuts following the results in Wu [39].

We illustrate the above results comparing two generic POSS strategies of players I and II in the following example.

Example 15. (Example 12 continued.) Consider the strategy $x^* = (0.7916, 0.2084)$ of player I obtained in the example (12). In order to get POSS strategies of player II we have to obtain the non-dominated solutions of the following multiple objective linear program:

$$\begin{aligned}(MLP_{II})_{\preceq} \min \quad & w_{11}, w_{12}, w_{21}\\ \text{s.t.} \quad & 175y_1 + 150y_2 \leq w_{11},\\ & 80y_1 + 175y_2 \leq w_{11},\\ & 190y_1 + 158y_2 \leq w_{12},\\ & 100y_1 + 190y_2 \leq w_{12},\\ & 180y_1 + 156y_2 \leq w_{21},\\ & 90y_1 + 180y_2 \leq w_{21},\\ & y_1 + y_2 = 1,\ y_1, y_2 \geq 0.\end{aligned}$$

Solving the above problem we choose the strategy $y^* = (0.2623, 0.7377)$ for player II being its security level $\widetilde{w^*} = (162.295, 5.738, 4.098)$. Note that $\widetilde{v^*} \preceq \widetilde{w^*}$ because the $\{0, 1\}$-cuts of $\widetilde{v^*}$ and $\widetilde{w^*}$ are

$$v^*_{\alpha_1} = [153.210, 164.670], \quad v^*_{\alpha_2} = 161,$$

$$w^*_{\alpha_1} = [156.557, 166.393], \quad w^*_{\alpha_2} = 162.295.$$

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References

[1] J.P. Aubin, Coeur et valeur des jeux flous a paiments lateraux, Comptes Rendus de l'Académie des Sciences, Paris Série A 279 (1974) 891–894.

[2] J.P. Aubin, Coeur et valeur des jeux flous sans paiments lateraux, Comptes Rendus de l'Académie des Sciences, Paris Série A 279 (1974) 963–966.

[3] C.R. Bector, S. Chandra, Fuzzy Mathematical Programming and Fuzzy Matrix Games, Springer-Verlag, 2005.

[4] C.R. Bector, S. Chandra, V. Vidyottama, Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs, Fuzzy Sets and Systems 146 (2004) 253–269.

[5] R. Branzei, N. Llorca, J. Sánchez-Soriano, S. Tijs, Multi-choice clan games and their core, TOP 17 (1) (2009) 123–138.

[6] L. Campos, Fuzzy linear programming models to solve fuzzy matrix games, Fuzzy Sets and Systems 32 (1989) 275–289.

[7] L. Campos, A. Gonzalez, M.A. Vila, On the use of the ranking function approach to solve fuzzy matrix games in a direct way, Fuzzy Sets and Systems 49 (2) (1992) 193–203.

[8] Y.-W. Chen, M. Larbani, Two-person zero-sum game approach for fuzzy multiple attribute decision making problems, Fuzzy Sets and Systems 157 (2006) 34–51.

[9] D. Dubois, E.E. Kerre, R. Mesiar, H. Prade, Fuzzy interval analysis, Fundamentals of Fuzzy Sets, vol. 1, Kluwer Academic Publishers, Dordrecht, 2000, pp. 483–581.

[10] D. Dubois, H. Prade, Operations on fuzzy numbers, International Journal of Systems Science 9 (1978) 613–626.

[11] D. Dubois, H. Prade, Fuzzy sets and systems. Theory and applications, Mathematics in Sciences and Engineering Series, vol. 144, Academic Press, New York, 1980.

[12] D. Dubois, H. Prade, Fundamentals of Fuzzy Sets, The Hand Books of Fuzzy Sets Series, Kluwer Academic Publishers, Boston, 2000.

[13] E. Fernández, S. Bernal, J. Navarro, R. Olmedo, An outranking-based fuzzy logic model for collaborative group preferences, TOP 18 (2) (2010) 444–464.

[14] F.R. Fernández, J. Puerto, Vector linear programming in zero-sum multicriteria matrix games, Journal of Optimization Theory and Applications 89 (1996) 115–127.

[15] F.R. Fernández, J. Puerto, M.J. Zafra, Cores of stochastic cooperative games, International Game Theory Review 4 (3) (2002) 265–280.

[16] J. Puerto, F.R. Fernández, Y. Hinojosa, Partially ordered cooperative games: extended core and shapley value, Annals of Operation Research 158 (2008) 143–159.
[17] F.R. Fernández, L. Monroy, J. Puerto, Multicriteria goal games, Journal of Optimization Theory and Applications 99 (2) (1998) 403–421.
[18] N. Furukawa, Parametric orders on fuzzy numbers and their roles in fuzzy optimization problems, Optimization 40 (2) (1997) 171–192.
[19] D. Ghose, A necessary and sufficient condition for Pareto-optimal security strategies in multicriteria matrix games, Journal of Optimization Theory and Applications 68 (3) (1991) 463–480.
[20] D. Ghose, R. Prassad, Solution concept in two-person multicriteria games, Journal of Optimization Theory and Applications 63 (2) (1989) 167–189.
[21] A. González, M.A. Vila, A discrete method for studying indifference and order relations between fuzzy numbers, Information Sciences 56 (1991) 245–258.
[22] A. González, M.A. Vila, Dominance relation on fuzzy numbers, Information Sciences 64 (1992) 1–16.
[23] F. Kacher, M. Larbani, Existence of equilibrium solution for a non-cooperative game with fuzzy goals and parameters, Fuzzy Sets and Systems 159 (2) (2008) 164–176.
[24] E.E. Kerre, X. Wang, Reasonable properties for the ordering of fuzzy quantities: part I and part II, Fuzzy Sets and Systems 118 (2001) 375–385, 387–405.
[25] M. Larbani, Existence of equilibrium solution for a non-cooperative game with fuzzy goals and parameters, Fuzzy Sets and Systems 159 (2) (2008) 164–176.
[26] M. Larbani, Solving bimatrix games with fuzzy payoffs by introducing nature as a third player, Fuzzy Sets and Systems 160 (5) (2009) 657–666.
[27] M. Larbani, Non cooperative fuzzy games in normal form: a survey, Fuzzy Sets and Systems 160 (22) (2009) 3184–3210.
[28] D.F. Li, A fuzzy multiobjective approach to solve fuzzy matrix games, The Journal of Fuzzy Mathematics 7 (1999) 873–880.
[29] T. Maeda, On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs, Fuzzy Sets and Systems 139 (2003) 283–296.
[30] M. Mares, Weak arithmetics of fuzzy numbers, Fuzzy Sets and Systems 91 (2) (1997) 143–154.
[31] I. Nishizaki, M. Sakawa, Fuzzy and Multiobjective Games for Conflict Resolution, Physica-Verlag, Heidelberg, 2001.
[32] G. Owen, Game Theory, second ed., Academic Press, New York, 1982.
[33] R.E. Steuer, Multiple Criteria Optimization: Theory, Computation and Application, Willey, New York, 1986.
[34] J. Ramík, J. Rímánek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, Fuzzy Sets and Systems 16 (2) (1985) 123–138.
[35] R.E. Steuer, Manual for the ADBASE. Multiple Objective Linear Programming Package, University of Georgia, Athens, Georgia, 1995.
[36] J. Suijs, P. Borm, A. De Waegenaere, Stochastic cooperative games in insurance, Insurance 22 (1998) 209–228.
[37] J. Suijs, P.P. Borm, A. De Waegenaere, S. Tijs, Cooperative games with stochastic payoffs, European Journal of Operational Research 113 (1999) 193–205.
[38] S. Tijs, R. Brânzei, Various characterizations of convex fuzzy games, TOP 12 (2) (2004) 399–408.
[39] H.-C. Wu, Duality theory in fuzzy linear programming problems with fuzzy coefficients, Fuzzy Optimization and Decision Making 2 (1) (2003) 61–73.
[40] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.